DATE OF EXAM November 17, 2016 **Solution**
SUBJECT NAME Optimization **Final Exam** **-** **Semester I**
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1. *(a) Let $A \in M_n(\mathbb{R})$ be invertible, u, v be column vectors in $\mathbb{R}_n$ and α be a real number. If $A + \alpha uv^t$ is invertible, show, by direct verification, that*

$$(A + \alpha uv^t)^{-1} = A^{-1} + \beta A^{-1} uv^t A^{-1},$$

for appropriate β. Hence, determine all the values of α for which $A + \alpha uv^t$ is invertible.

Solution: We claim that $A + \alpha uv^t$ is invertible if $1 + \alpha v^t A^{-1} u \neq 0$ and $A^{-1} + \beta A^{-1} uv^t A^{-1}$ is the inverse of $A + \alpha uv^t$, where $\beta = -\frac{\alpha}{1+\alpha v^t A^{-1} u}$.

$$(A + \alpha uv^t) \left(A^{-1} - \frac{\alpha A^{-1} uv^t A^{-1}}{1 + \alpha v^t A^{-1} u} \right)$$

$$= I + \alpha uv^t A^{-1} - \frac{\alpha u(1 + \alpha v^t A^{-1} u) v^t A^{-1}}{1 + \alpha v^t A^{-1} u}$$

$$= I.$$

Hence the claim.

(b) Let e_j, $1 \leq j \leq n$, denote the standard unit vectors in $\mathbb{R}^n$. Put $E_{ij} = e_i e_j^t$ for $1 \leq j \leq n$. By using (a) above or otherwise, find all real λ and μ such that the matrix $I + \lambda E_{1n} + \mu En1$ is invertible. Find an expression for the inverse in that case.

Solution: First note that if any one of λ or μ is zero then the inverse of $I + \lambda E_{1n}$ is $I + -\lambda E_{1n}$ and the inverse of $I + \mu E_{1n}$ is $I + -\mu E_{1n}$. Now let both λ and μ be non-zero and let $A = I + \lambda E_{1n}$. Then applying A, we see that $I + \lambda E_{1n} + \mu E_{n1}$ has an inverse if $1 + \mu e_1^t (I - \lambda E_{1n} e_n^t) \neq 0$, that is $\lambda\mu \neq 1$.

2. Using the singular value decomposition or otherwise, prove the following:

(a) Suppose A, B are real $m \times n$ matrices such that $A^t A = B^t B$. Show that there is an orthogonal $m \times m$ matrix U such that $A = UB$.

Solution: Since $A^t A = B^t B$, it follows that the singular value decomposition for A and B are respectively,

$$A = VDU^T \text{ and } B = V'DU^t$$

for some orthogonal $n \times n$ matrix U and orthogonal $m \times m$ matrices V, V'. Then $V(V')^t B = A$ and $V(V')^t$ is an $m \times m$ orthogonal matrix.

(b) Suppose $A \in M_n(\mathbb{R})$. Show that the eigenvalues of A^tA and AA^t are the same and with the same algebraic multiplicities.

Solution: Note that $(A^tA)^t = AA^t$. Let $B = A^tA$, then $(B - \lambda I)^t = B^t - \lambda I$ and, therefore, $\det(B - \lambda I) = \det(B^t - \lambda I)$. So, A^tA and AA^t has the smae characteristic equation. Hence they have the same eigenvalues with same algebraic multiplicities.

3. *(a) Let $A = ((a_{ij})) \in M_n(\mathbb{R})$ be a positive matrix, that is $a_{ij} > 0$ for all i, j. Suppose there is a $\lambda > 0$ and a vector $x \geq 0$, $x \neq 0$ such that $Ax \geq \lambda x$, $Ax \neq \lambda x$. Show that there is a vector $y \geq 0$, $y \neq 0$ such that $Ay > \lambda y$.*

Solution: Consider $A - \lambda I$ as a continuous function from $\mathbb{R}^n$ to $\mathbb{R}^n$. Then for any w in $\mathbb{R}^n$,

$$(A - \lambda I)(w) = (f_1(w), ..., f_n(w)),$$

where each f_i, $1 \leq i \leq n$, is a continuous function from $\mathbb{R}^n$ to $\mathbb{R}$. By the given hypothesis, $x \in \mathbb{R}^n$ is such that $f_i(x) > 0$ for f_i, $1 \leq i \leq n$. Let $\delta > 0$ be such that $f_i(x) > \delta$ for $1 \leq i \leq n$. Then $f_i^{-1}(\delta, \infty)$ is an open set in $\mathbb{R}^n$ containing x for all i, $1 \leq i \leq n$. Then $\bigcap_{i=1}^n f_i^{-1}(\delta, \infty)$ is a non-empty open set containing x. Hence, there exists $y \geq 0$, $y \neq 0$ such that $Ay > \lambda y$.

(b) Suppose $A \in M_n(\mathbb{R})$ is a non-negative, irreducible matrix. Let $\rho = \rho_A$ be the the eigenvalue of A such that ρ equals the spectral radius of A. If x is a non-zero real or complex n-vector such that $(A - \rho I)^2 x = 0$, show that $(A - \rho I)x = 0$.

Solution: Let x be such that $(A - \rho I)^2 x = 0$. So, $(A - \rho I)x \in \ker(A - \rho I) = \operatorname{span}\{u\}, u > 0$. Therefore, $(A - \rho I)x = cu$. If $c = 0$, then we are done. So, suppose $c \neq 0$. Since A is real and $\rho > 0$, we may choose x to be a real vector. Also by changing x to $-x$, we may assume that $c > 0$. Then $Ax = cu + \rho x$, $c.0, u > 0$.

Now, for $h \in \mathbb{R}$, consider the vector $x + hu$.

$$A(x + hu) = Ax + hAu = cu + \rho x + h\rho u = cu + \rho(x + hu).$$

By choosing h appropriately, we can make sure that $x + hu > 0$, then $A(x + hu) > \rho(x + hu)$, which is a contradiction to the definition of ρ. Hence $(A - \rho I)x = 0$.

4. *(a) Show that 1 is the dominant eigenvalue of the doubly stochastic matrix*

$$A = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \end{pmatrix}$$

and hence find the limit $\lim_{k \to \infty} A^k$.

Solution: Let $C = \{x \in \mathbb{R}^4 : x \geq 0\}$ and $B = \{x \in \mathbb{R}^4 : ||x|| = 1\}$. Then we know that $\delta_A(x) \leq \max_i \sum\limits_{j=1}^{n} a_{ij}$ for $x \in B \cap C$. Therefore, in this case $\delta_A(x) \leq 1$, for $x \in B \cap C$. Therefore, $\rho_A \leq 1$. But 1 is an eigenvalue of A and $(1, ..., 1)$ is an eigenvector corresponding to the eigenvalue 1. Therefore, $\rho_A = 1$ and 1 is the dominant eigenvalue. Same is true for A^t as well. Hence,

$$\lim_{k \to \infty} A^k = (1, ..., 1)^t(1, ..., 1) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

(b) Let T and S be non-negative matrices such that T is irreducible and $T - S$ is non-negative. Prove that $spr(T) \geq spr(S)$ and the equality occurs only if $T = S$.

Solution: Let $\beta \in spec(S)$ be such that $|\beta| = spr(S)$. Let $y = (y_1, ..., y_n)^t \in \mathbb{R}^n$ be such that $Sy = \beta y$. Let $v = (v_1, ..., v_n)^t = (|y_1|, ..., |y_n|)^t$, then

$$|\beta|v_i = |\beta y_i| = \left| \sum_{j=1}^{n} S_{ij} y_j \right| \leq \sum_{j=1}^{n} S_{ij} v_j \leq \sum_{j=1}^{n} T_{ij} v_j, \tag{1}$$

which implies that

$$spr(S) = |\beta| \leq \delta_T(v) \leq spr(T),$$

where $\delta_T(v) = min_{1 \leq i \leq n} \left\{ \frac{<Tv, e_i>}{<v, e_i>} :< v, e_i >> 0 \right\}$, e_i denoting the ith member of the standard basis of $\mathbb{R}^n$.

Now let $spr(S) = spr(T)$, we will show that $S = T$. From (5) we have $T(v) - spr(S)v \geq 0$ and therefore, $T(v) - spr(T)v \geq 0$ since $spr(S) = spr(T)$. If $T(v) - spr(T)v > 0$, then

$$< Tv, e_i >> spr(T) < v, e_i > .$$

This implies that $\delta_T(v) > spr(T)$, which is a contradiction. Hence $T(v) = spr(T)v$. Since T is also irreducible, it follows that $v_i > 0$ for $1 \leq i \leq n$.

Since $\beta v = Tv$ and $v_i > 0$, we get

$$\sum_{j=1}^{n} T_{ij} v_j = spr(T) v_i = spr(S) v_i = \sum_{j=1}^{n} S_{ij} v_j,$$

which implies that $\sum_{j=1}^{n} (T_{ij} - S_{ij}) v_j = 0$. Hence $S = T$.

5. (a) Solve the following using simplex method:

$$\begin{aligned} &\text{minimize } 5x_1 - 8x_2 - 3x_3 \\ &\text{subject to } 2x_1 + 5x_2 - x_3 \leq 1 \\ &\qquad -3x_1 - 8x_2 + 2x_3 \leq 4 \\ &\qquad -2x_1 - 12x_2 + 3x_3 \leq 9 \\ &\qquad\qquad x_i \geq 0,\ i = 1, 2, 3. \end{aligned}$$

Solution: The initial simplex tabeulo is

x_1	x_2	x_3	x_4	x_5	x_6	z	
2	5	-1	1	0	0	0	1
-3	-8	2	0	1	0	0	9
-2	12	3	0	0	1	0	4
-5	8	3	0	0	0	1	0

after doing the row operations: $R_2 + \frac{3}{2}R_1$, $R_3 + R_1$, $R_4 + \frac{5}{2}R_1$ and $\frac{1}{2}R_1$, we get,

x_1	x_2	x_3	x_4	x_5	x_6	z	
1	$\frac{5}{2}$	$\frac{-1}{2}$	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$
0	$\frac{-1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	1	0	0	$\frac{11}{2}$
0	-7	2	1	0	1	0	10
0	$\frac{41}{2}$	$\frac{1}{2}$	$\frac{5}{2}$	0	0	1	$\frac{5}{2}$

From which we conclude that the optimal solution is given by $x_1 = \frac{1}{2}$, $x_2 = 0$, $x_3 = 0$ and the optimum value is $\frac{5}{2}$.

(b) Find the dual of the linear programming

$$\textit{maximize } c^t x, \textit{ subject to } Ax \leq b,\ x \geq 0,$$

where A is an $m \times n$ real matrix and b, c are given column vectors in $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. In this set up, state and prove the weak duality lemma.

Solution: The dual problem is given by

$$\text{minimize } b^t y, \text{ subject to } A^t y \geq c,\ y \geq 0.$$

The weak duality lemma states that if x is a feasible solution to the primal problem and y is a feasible solution to the dual problem, then $c^t x \leq b^t y$. The proof goes as follows:

$$c^t x = x^t c \leq x^t (A^t y) \ \text{(since } y \text{ is a feasible solution to the dual problem)}$$
$$= (Ax)^t y \leq b^t y \ \text{(since } x \text{ is a feasible solution to the primal problem)} .$$

6. *(a) Let A be an $m \times n$ real matrix, whose rank is m; b, c are given column vectors in $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. Consider the following linear programming:*

$$\textit{minimize } c^t x, \textit{ subject to } Ax = b,\ x \geq 0.$$

If $A = (a_1 ... a_n)$, assume that the matrix $B = (a_1 ... a_m)$ is non-singular and that there is a vector $x \in \mathbb{R}^m$, $x \geq 0$ satisfying $Bx = b$.

for $\varepsilon > 0$ small consider the system $Ax = b(\varepsilon)$, where $b(\varepsilon) = b + \varepsilon a_1 + ... + \varepsilon^n a_n$. Show that there is a vector $y \in \mathbb{R}^m$, $y > 0$ satisfying $By = b(\varepsilon)$.

Solution: Let $x = (x_1, ... x_m)^t$ and $x' = (x_1, ..., x_m, 0, ..., 0)^t + (\varepsilon, ..., \varepsilon^n)$. Then $Ax' = b(\varepsilon)$ Since $a_1, ..., a_n$ are linearly dependent there exist $y_1, ..., y_n$ such that $y_1 a_1 + ... + y_n a_n = 0$ and, therefore,

$$(x_1 + \varepsilon - \delta y_1)a_1 + ... + (x_m + \varepsilon^m - \delta y_m)a_m + (\varepsilon^{m+1} - \delta y_{m+1})a_{m+1} + ... + (\varepsilon^n - \delta y_n)a_n = b(\varepsilon),$$

for any δ. Now choosing $\delta = min_i\{\frac{x_i + \varepsilon^i}{y_i} : y_i > 0\}$, we can obtain another feasible solution which is a linear combination of at most $n-1$ of the vectors a_i's. Note that choosing ε appropriately (that's why the appropriate range of ε comes into play), we can make sure that the coefficients of at least m of the vectors a_i's are positive. Continuing this way after finitely many steps we obtain a basic feasible solution $y' > 0$, which gives rise to the necessary vector $y \in \mathbb{R}^m$, $y > 0$ such that $By = b(\varepsilon)$.

(b) Consider the linear programme (P) of the form

$$\textit{minimize } q^t z, \textit{ subject to } Mz \geq -q,\ z \geq 0$$

where $M \in M_k(\mathbb{R})$ *is skew symmetric matrix and* $q \in \mathbb{R}^k$*. Show that the problem* (P) *and its dual are the same. Further, show that any feasible solution of* (P) *is also optimal.*

Solution: The dual problem (D) of (P) is given by

$$\text{maximize } -q^t z, \text{ subject to } M^t z \leq q,\ z \geq 0$$

which is same as

$$\text{minimize } q^t z, \text{ subject to } Mz \geq -q,\ z \geq 0,$$

since M is skew symmetric.

Now let x be a feasible solution of (P), and let y be another feasible solution of (P). Then y is also a feasible solution of the dual problem (D). Then by weak duality theorem we know that $q^t x \leq q^t y$. Hence x is an optimal solution of (P).

7. *(a) Suppose* C *is a convex set in* $\mathbb{R}^n$ *and* $k \geq 2$*. Let* $x_1, ..., x_k$ *are in* C *and* $t_1, ..., t_k$ *are non-negative real numbers such that* $t_1 + ... + t_k = 1$*. Show that* $t_1x_1 + ... + t_kx_k$ *is also in* C*.*

Solution: Suppose $k = 3$. Note that

$$t_1x_1 + t_2x_2 + t_3x_3 = (t_1 + t_2)\left(\frac{t_1}{t_1 + t_2}x_1 + \frac{t_2}{t_1 + t_2}x_2\right) + t_3x_3.$$

Now by convexity of C, $x' = \frac{t_1}{t_1+t_2}x_1 + \frac{t_2}{t_1+t_2}x_2 \in C$ and

$$t_1x_1 + t_2x_2 + t_3x_3 = (t_1 + t_2)x' + t_3x_3 \in C.$$

The rest of the arguement can be completed easily using induction.

(b) Define an extreme point of a convex set in $\mathbb{R}^n$*.*

Solution: A point x in a convex set C is called an extreme point if whenever $x = ty + (1 - t)z$ for $y, z \in C$ and $0 < t < 1$, we have $x = y = z$.

(c) Consider the set of constraints $Ax = b,\ x \geq 0$*, where* A *is a real* $m \times n$ *matrix of rank* m *and* $b \in \mathbb{R}^m$*.*
i. Define a basic feasible solution of the above set of constraints.
ii. State and prove the result concerning the extreme points of the set of all feasible solutions of the above set of constraints.

Solution: *i.* Let $A = (a_1...a_n)$, a_j is the jth column of A. Assume $a_1, ...a_m$ are linearly independent and $B = (a_1...a_m)$. then there is a unique $x_B \in \mathbb{R}^m$ such that $Bx_B = b$. If in addition $x_B \geq 0$, then $x = \begin{pmatrix} x_B \\ 0 \\ . \\ . \\ 0 \end{pmatrix} \in \mathbb{R}^n$ satisfies $Ax = b$. This $x = \begin{pmatrix} x_B \\ 0 \\ . \\ . \\ 0 \end{pmatrix}$ is called a basic feasible solution.

ii. **Statement:** Let K be the set of feasible solutions of the above mentioned constraints. Then the extreme points of the convex set K are precisely the basic feasible solutions.
Proof: Suppose $x = (x_1, ..., x_m, 0, ..., 0)$ is a basic feasible solution. Then

$$x_1 a_1 + ... + x_m a_m = b.$$

Suppose y and z are feasible solutions and $0 < t < 1$ such that $x = ty + (1-t)z$. Then it is clear that $y_j = z_j = 0$ for $j > m$. Therefore,

$$y_1 a_1 + ... + y_m a_m = b, \quad z_1 a_1 + ... + z_m a_m = b.$$

Then by linear independence of $a_1, ..., a_m$, it follows that $y_i = z_i$ for $1 \leq i \leq m$. Hence $y = z$ and consequently, $x = y = z$. So, x is an extreme point of K.

Conversely suppose that x is not a basic feasible solution. It follows from the proof of the fundamental theorem of linear programming that for sufficiently small $\varepsilon > 0$, there is a feasible solution $y \neq 0$ such that $x + \varepsilon y$ and $x - \varepsilon y$ both are feasible solutions. Then

$$x = \frac{1}{2}(x + \varepsilon y) + \frac{1}{2}(x - \varepsilon y).$$

Hence x is not an extreme point of K. This completes the proof.